**Calculus Application Problem - Heat It, Then Cool It!**

**Introduction:** A temperature probe is placed in a cup of hot water. It remains there for approximately 35 seconds, then it is removed from this cup and placed in a cup of cold water for another 35 seconds. The objective of this activity is to find an algebraic function that models the temperature recorded by the probe over the entire 70 seconds, then apply some calculus concepts to the function.

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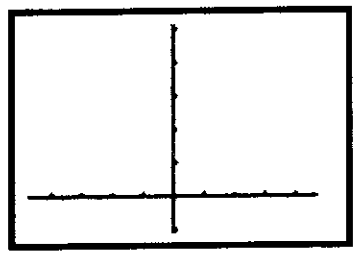
**An Algebra Review of Piecewise Functions:**

A piecewise function is a function that is defined differently over different values of the domain. Two examples are presented. In the first, a piecewise function is given, and you are to graph it. In the second, the graph of a piecewise function is shown, and you are to write its rule.

1. Given the function $f(x)=\begin{cases} -x+2 & \text{if } x\le 1 \\ \sqrt{x} & \text{if } x>1 \end{cases}$

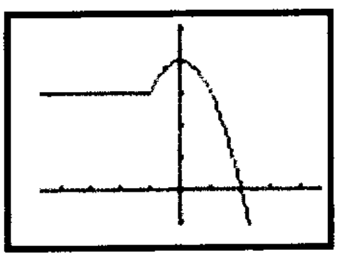


   a. Evaluate: f(0) = _______

      f(4) = _______

   b. Sketch the graph of f.

2. Write a rule for the piecewise function shown below.



$f(x)=\begin{cases} \\ \\ \end{cases}$

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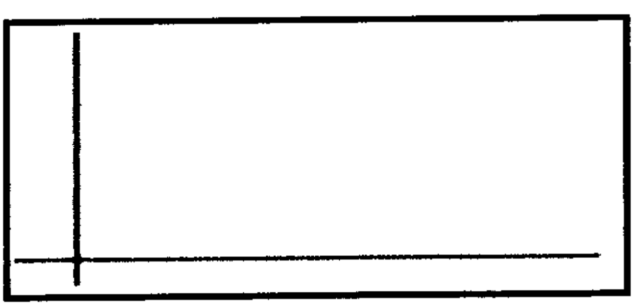
1. Complete the chart below from the data that was collected from the experiment. The data appears to have **horizontal asymptotes** at y = ____________ and y = ____________

| **Time** | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Temp** | | | | | | | | | | | | | | | |

2. Create a scatterplot of the data on your calculator (**Time** in **L1** and **Temperature** in **L2**). Sketch the scatterplot below.



3. **The General Solution (Newton's Law of Cooling/Heating)**

We want to find a function **F(t)** that models the Fahrenheit temperature **F** of the probe at any time **t**, measured in seconds. Using a property of physics, called **Newton's Law of Cooling/Heating**, the temperature in an activity such as this can be modeled by an exponential function in the form:

$$\mathbf{F(t) = a \cdot b^t + c}.$$

4. **The Specific Solution**

First, to find an algebraic function that models our temperature vs. time data, it is clear that we need to write a piecewise function, one rule for the first (approximately) 35 seconds, and another for the last 35 (approximately) seconds. To find both of these rules, we will use the property of Newton's Law of Cooling.

In order to find a model of the form $\mathbf{F(t) = a \cdot b^t + c}$ for the first part of our data, we need to find the constants **a**, **b**, and **c**. We can find the constants **a** and **c** from the chart above (or by "tracing" on our scatterplot).

First we will find the constant **c**. According to Newton's Law of Cooling/Heating, and the data collected, the value of **c** would be approximately ____________.

To find **a**, record the temperature when $t = 0$. ( 0 , ________ ) Substitute this ordered pair, with the value of **c**, into our model $\mathbf{F(t) = a \cdot b^t + c}$, and solve for **a**. Show your work below.

**a=** ____________

To find **b**, the last constant in the model, we need another ordered pair. Let's use the ordered pair from the chart when $t = 10$ seconds.

Record this ordered pair. ( 10 , ________ )

Substitute these values into the equation (with the values of **a** and **c**), and solve for the last unknown constant **b**. Show your work below. Round **b** to three decimal places. Then write the function **F(t)**.

**b=** ____________

**F(t)=** ________________

To check your work, graph your equation with your scatterplot to see how it fits the first part of the data. If it doesn't fit well, find your mistake!

The equation that fits the second part of the data is similar to the first equation. However, since we are beginning with a time other than $t = 0$, we need to apply a "horizontal shift" to our function. Therefore, the resulting form of this function is $\mathbf{F(t) = a \cdot b^{t-h} + c}$, where $\mathbf{h = 35}$. Use this form, and the hints given for the first part of the function, and find a rule that fits this part of the data in the scatterplot. Show all of your work below. Again, graph this equation to check it.

**a=** ____________

**b=** ____________ (3 decimals)

**c=** ____________

**F(t)=** ____________________

$$F'(t) = \begin{cases} \\ \\ \end{cases}$$

Verify your answer by graphing it on your calculator and comparing it to the graph drawn in part b above.

d. Using the function **F'(t)** above, how fast is the temperature changing when **t = 0, t = 10, and t = 45**? State the units with your answer.

**F'(0)** = ______________ **F'(10)** = ______________ **F'(45)** = ______________

e. An important calculus property, called the **Mean Value Theorem** (used in proving many calculus properties), says that on the domain of a function where the graph is "smooth"; i.e. no breaks or cusps, there must exist "at least one t-value where the instantaneous rate of change is equal to the average rate of change".

i. Consider the the time interval [ 0 , 15 ]. What is the average rate of change of **F(t)** on this interval? Show the expression below that you are using in evaluating this value.

ii. Since the derivative of a function represents the instantaneous rate of change of the function at any t-value, set the derivative **F'(t)** equal to the value found in part i above, and solve for **t**. Show your work!

iii. Now consider the the time interval [ 40 , 50 ]. What is the average rate of change of **F(t)** on this interval? Show the expression below that you are using in evaluating this value.

iv. Find the t-value where the instantaneous rate of change is equal to the average rate of change on [ 40 , 50 ]. Show your work!